NASA-TM-111554

Nondimensional Parameters and Equations for Buckling of Anisotropic Shallow Shells

M. P. Nemeth
Senior Research Engineer,
Structural Mechanics Division,
NASA Langley Research Center,
Hampton, VA 23681-0001

A procedure for deriving nondimensional parameters and equations for bifurcation buckling of anisotropic shallow shells subjected to combined loads is presented. First, the Donnell-Mushtari-Vlasov equations governing buckling of symmetrically laminated doubly curved thin elastic shallow shells are presented. Then, the rationale used to perform the nondimensionalization of the buckling equations is presented, and fundamental parameters are identified that represent measures of the shell orthotropy and anisotropy. In addition, nondimensional curvature parameters are identified that are analogues of the well-known Batdorf Z parameter for isotropic shells, and analogues of Dunnell's and Batdorf's shell buckling equations are presented. Selected results are presented for shear buckling of balanced symmetric laminated shells that illustrate the usefulness of the nondimensional parameters.

Introduction

Understanding the buckling behavior of anisotropic shallow shells made of laminated composite materials is important for the structural design of future high performance aircraft. Identifying fundamental parameters that characterize buckling behavior of shells for a wide range of laminate configurations and materials will greatly aid the preliminary design of aircraft components such as the fuselage and empennage. More specifically, nondimensional parameters permit results to be presented as a series of curves, on one or more plots, that span a wide range of shell dimensions, loading combinations, boundary conditions, and material properties. Design charts of this general nature allow for quick evaluation of several design alternatives and furnish the designer with insight into the sensitivity of a particular design to changes in geometry, loading conditions, boundary conditions, or material properties. Of equal importance is the potential for nondimensional parameters to provide insight into the development of scaling laws for composite shells that will be valuable for relating subscale tests to full-scale tests during the certificiation phase of aircraft design. Potentially, well-defined scaling laws would minimize the amount of full-scale structural testing by using a series of less expensive subscale tests to complement the full-scale tests.

Characterizing the buckling behavior of anisotropic shells is not a trivial task due to the complex deformational coupling they exhibit. However, there are classes of practical laminated composite shells that do not exhibit total deformational coupling. The present study focuses on one such class of shells commonly known as symmetrically laminated shells. These shells exhibit anisotropy in the form of material-induced coupling between pure bending and twisting deformations and coupling between pure biaxial stretching and membrane shearing.

The objectives of the present paper are to present a method of deriving nondimensional equations for doubly curved anisotropic shallow shells subjected to combined loads, and to identify the fundamental parameters associated with bifurcation buckling of these shells. Shells with a high degree of curvature are known to be sensitive to small imperfections in their geometry under certain loading conditions, and this imperfection sensitivity leads to collapse loads often substantially lower than a predicted bifurcation buckling load (Almroth and Brogan, 1972). However, a class of shallow shells exists for which imperfection sensitivity is minimal under certain loading conditions (Stein and McElman, 1965). In this case, results obtained from a bifurcation buckling analysis can be used to obtain credible estimates of the collapse load. Moreover, in studying the general collapse behavior of shells, the researcher is often interested in knowing the bifurcation-type response for the sake of comparison. Furthermore, the parameters identified in a simpler bifurcation shell buckling analysis may be adequate for use in characterizing the actual nonlinear collapse behavior. A significant example of this approach is the well-known nondimensional Batdorf Z parameter that has seen wide use in buckling analyses of isotropic cylindrical shells (Gerard, 1962).

The present paper begins with a presentation of the Donnell-Mushtari-Vlasov equations governing bifurcation buckling of shallow shells that are thin, symmetrically laminated, and elastic. The equations are put into a form suitable for nondimensionalization, and the rationale used to perform the nondimensionalization of the buckling equations is presented.

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for publication in the ASME Journal of Applied Mechanics.

Discussion on this paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME Journal of Applied Mechanics.

Manuscript received by the ASME Applied Mechanics Division, Dec. 12, 1992; final revision, Apr. 12, 1993. Associate Technical Editor: X. Markenscoff.

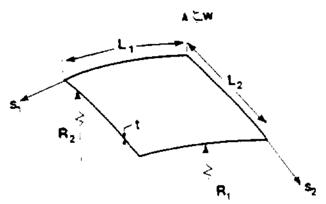

Fig. 1 Geometry of a shallow shell

Fundamental parameters are then identified that represent measures of the membrane and bending orthotropy and anisotropy. Curvature parameters for symmetrically laminated shells are then presented that are analogues of the Batdorf Z parameter. Next, analogues to Donnell's and Batdorf's shell buckling equations are presented. Last, selected results are presented for shear buckling of balanced symmetrically laminated shells to illustrate the usefulness of the nondimensional parameters.

The analysis presented in the present paper was inspired by the work presented by Stein (1982). For this reason, and for many useful discussions on this subject, the author would like to dedicate this paper to the late Dr. Manuel Stein of NASA Langley Research Center who spent nearly 50 years studying the buckling behavior of plates and shells.

Equations Governing Buckling

Equations governing buckling of thin elastic doubly curved shallow shells have been derived by Nemeth (1991) from the Donnell-Mushtari-Vlasov nonlinear equations in terms of lines-of-curvature coordinates using the method of adjacent equilibrium states. In the derivation, prebuckling displacements normal to the surface are neglected and the lines-of-curvature coordinates s_1 and s_2 with units of length are used to simplify the form of the equations (see Fig. 1). The use of two arc-length coordinates to parametrize the shell reference surface, which is not developable in general, is consistent with the assumptions of shallow shell theory that the surface metric tensor equals the Kronecker delta function and that the surface compatibility equation of Gauss is approximately satisfied. The linearized equilibrium equations are given by

$$\frac{\partial N_1}{\partial s_1}+\frac{\partial N_{12}}{\partial s_2}=0 \tag{1a}$$

$$\frac{\partial N_{12}}{\partial s_1}+\frac{\partial N_2}{\partial s_2}=0 \tag{1b}$$

$$\frac{\partial^2 M_1}{\partial s_1^2}+2\frac{\partial^2 M_{12}}{\partial s_1 \partial s_2}+\frac{\partial^2 M_2}{\partial s_2^2}-\frac{N_1}{R_1}-\frac{N_2}{R_2} + N_1^0\frac{\partial^2 w}{\partial s_1^2}+N_2^0\frac{\partial^2 w}{\partial s_2^2}+2N_{12}^0\frac{\partial^2 w}{\partial s_1 \partial s_2}=0 \tag{1c}$$

where N_1, N_2, and N_{12} are the membrane stress resultants and M_1, M_2, and M_{12} are the bending stress resultants of the adjacent equilibrium state. The quantities N_1^0, N_2^0, and N_{12}^0 are the membrane stress resultants of the primary equilibrium state and are referred to herein as the prebuckling stress resultants. The quantity w is the normal deflection of the shell at the onset of buckling and is referred to herein as the buckling displacement. The symbols R_1 and R_2 denote the principal radii of curvature of the shell middle surface along the s_1 and s_2 coordinate directions, respectively. Similarly, the linearized buckling compatibility equation is given by

$$\frac{\partial^2 \epsilon_1}{\partial s_2^2}+\frac{\partial^2 \epsilon_2}{\partial s_1^2}-\frac{\partial^2 \gamma_{12}}{\partial s_1 \partial s_2}=\frac{1}{R_2}\frac{\partial^2 w}{\partial s_1^2}+\frac{1}{R_1}\frac{\partial^2 w}{\partial s_2^2} \tag{2}$$

where ϵ_1, ϵ_2, and γ_{12} are the membrane strains of the adjacent equilibrium state. The constitutive equations for a symmetrically laminated anisotropic shell are given by

$$N_1=A_{11}\epsilon_1+A_{12}\epsilon_2+A_{16}\gamma_{12} \tag{3a}$$

$$N_2=A_{12}\epsilon_1+A_{22}\epsilon_2+A_{26}\gamma_{12} \tag{3b}$$

$$N_{12}=A_{16}\epsilon_1+A_{26}\epsilon_2+A_{66}\gamma_{12} \tag{3c}$$

$$M_1=-D_{11}\frac{\partial^2 w}{\partial s_1^2}-D_{12}\frac{\partial^2 w}{\partial s_2^2}-2D_{16}\frac{\partial^2 w}{\partial s_1 \partial s_2} \tag{4a}$$

$$M_2=-D_{12}\frac{\partial^2 w}{\partial s_1^2}-D_{22}\frac{\partial^2 w}{\partial s_2^2}-2D_{26}\frac{\partial^2 w}{\partial s_1 \partial s_2} \tag{4b}$$

$$M_{12}=-D_{16}\frac{\partial^2 w}{\partial s_1^2}-D_{26}\frac{\partial^2 w}{\partial s_2^2}-2D_{66}\frac{\partial^2 w}{\partial s_1 \partial s_2} \tag{4c}$$

where A_{11}, A_{12}, A_{22}, and A_{66} are the orthotropic membrane stiffnesses; A_{16} and A_{26} are the anisotropic membrane stiffnesses; D_{11}, D_{12}, D_{22}, and D_{66} are the orthotropic bending stiffnesses; and D_{16} and D_{26} are the anisotropic bending stiffnesses of classical laminated thin-shell theory (Dong, Pister, and Taylor, 1962). The corresponding boundary conditions of the boundary value problem are homogeneous and are not listed herein for brevity.

Nondimensionalization Procedure and Parameters

Nondimensional parameters for the buckling of shallow shells are obtained by building upon the procedure presented by Batdorf (1947*a–c*) for isotropic curved plates, by Stein (1982) for flat specially orthotropic laminated plates, and by Nemeth (1986) for flat symmetrically laminated plates. The basic premise of the procedure is to make the field variables and their derivatives of order one, to minimize the number of independent parameters required to characterize the behavior, and to avoid introducing a preferential direction into the nondimensional equations.

The first step in the nondimensionalization procedure is to formulate the boundary valve problem in terms of a single equilibrium equation and a compatibility equation. This step yields two coupled homogeneous linear partial differential equations. To obtain these equations, the membrane stress resultants of the adjacent equilibrium state are expressed in terms of a stress function Φ by

$$N_1=\frac{\partial^2 \Phi}{\partial s_2^2} \tag{5a}$$

$$N_2=\frac{\partial^2 \Phi}{\partial s_1^2} \tag{5b}$$

$$N_{12}=-\frac{\partial^2 \Phi}{\partial s_1 \partial s_2} \tag{5c}$$

This stress function satisfies Eqs. (1*a*) and (1*b*) identically. Eliminating these two equations from the boundary value problem by introducing a stress function requires that the buckling compatibility equation given by Eq. (2) be satisfied. To put the buckling compatibility equation into a convenient form, the inverted form of constitutive Eqs. (3*a*) through (3*c*) is used to express the buckling strains in terms of the stress function; i.e.,

$$\epsilon_1=a_{11}\frac{\partial^2 \Phi}{\partial s_2^2}+a_{12}\frac{\partial^2 \Phi}{\partial s_1^2}-a_{16}\frac{\partial^2 \Phi}{\partial s_1 \partial s_2} \tag{6a}$$

$$\epsilon_2=a_{12}\frac{\partial^2 \Phi}{\partial s_2^2}+a_{22}\frac{\partial^2 \Phi}{\partial s_1^2}-a_{26}\frac{\partial^2 \Phi}{\partial s_1 \partial s_2} \tag{6b}$$

$$\gamma_{12}=a_{16}\frac{\partial^2 \Phi}{\partial s_2^2}+a_{26}\frac{\partial^2 \Phi}{\partial s_1^2}-a_{66}\frac{\partial^2 \Phi}{\partial s_1 \partial s_2} \tag{6c}$$

where a_{11}, a_{12}, a_{22}, a_{16}, a_{26}, and a_{66} are the membrane flexibility coefficients of the shell and are functions of the membrane stiffnesses only. Substituting Eqs. (6) into Eq. (2) converts the buckling compatibility equation into an equation in terms of the stress function and buckling displacement. Following Stein (1982), the following nondimensional coordinates are used:

$$S_1 = s_1/L_1 \text{ and } S_2 = s_2/L_2 \tag{7}$$

where L_1 and L_2 are characteristic dimensions of the shell shown in Fig. 1. Substituting these nondimensional coordinates into the buckling compatibility equation and normalizing by a weighted geometric mean flexibility; i.e., multiplying through by $L_1^2L_2^2/\sqrt{a_{11}a_{22}}$, yields

$$\alpha_m^2 \frac{\partial^4\Phi}{\partial S_1^4} + 2\alpha_m\gamma_m \frac{\partial^4\Phi}{\partial S_1^3\partial S_2} + 2\mu \frac{\partial^4\Phi}{\partial S_1^2\partial S_2^2} + 2\frac{\delta_m}{\alpha_m}\frac{\partial^4\Phi}{\partial S_1\partial S_2^3} + \frac{1}{\alpha_m^2}\frac{\partial^4\Phi}{\partial S_2^4} = \frac{1}{\sqrt{a_{11}a_{22}}}\left[\frac{L_2^2}{R_2}\frac{\partial^2 w}{\partial S_1^2} + \frac{L_1^2}{R_1}\frac{\partial^2 w}{\partial S_2^2}\right] \tag{8}$$

where the nondimensional parameters appearing in the equation are given by

$$\alpha_m = \frac{L_2}{L_1}(a_{22}/a_{11})^{1/4} \tag{9a}$$

$$\mu = \frac{2a_{12}+a_{66}}{2\sqrt{a_{11}a_{22}}} \tag{9b}$$

$$\gamma_m = -\frac{a_{26}}{(a_{11}a_{22}^3)^{1/4}} \tag{9c}$$

and

$$\delta_m = -\frac{a_{16}}{(a_{11}^3a_{22})^{1/4}}. \tag{9d}$$

Expressions for these parameters written in terms of shell membrane stiffness coefficients are presented in the Appendix.

The next step in the nondimensionalization procedure is to express the normal-direction equilibrium equation in terms of the buckling displacement w and stress function Φ by substituting Eqs. (4) and (5) into Eq. (1c). Performing this step, introducing the nondimensional coordinates, and normalizing by a weighted geometric mean stiffness; i.e., multiplying through by $L_1^2L_2^2/\sqrt{D_{11}D_{22}}$, yields

$$\alpha_b^2 \frac{\partial^4 w}{\partial S_1^4} + 4\alpha_b\gamma_b \frac{\partial^4 w}{\partial S_1^3\partial S_2} + 2\beta \frac{\partial^4 w}{\partial S_1^2\partial S_2^2} + 4\frac{\delta_b}{\alpha_b}\frac{\partial^4 w}{\partial S_1\partial S_2^3} + \frac{1}{\alpha_b^2}\frac{\partial^2 w}{\partial S_2^4} + \frac{L_1^2}{R_1\sqrt{D_{11}D_{22}}}\frac{\partial^2\Phi}{\partial S_2^2} + \frac{L_2^2}{R_2\sqrt{D_{11}D_{22}}}\frac{\partial^2\Phi}{\partial S_1^2} - K_1\pi^2\frac{\partial^2 w}{\partial S_1^2} - K_2\pi^2\frac{\partial^2 w}{\partial S_2^2} - 2\frac{K_{12}\pi^2}{\alpha_b}\frac{\partial^2 w}{\partial S_1\partial S_2} = 0 \tag{10}$$

where the nondimensional parameters appearing in the equation are given by

$$\alpha_b = \frac{L_2}{L_1}(D_{11}/D_{22})^{1/4} \tag{11a}$$

$$\beta = \frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}} \tag{11b}$$

$$\gamma_b = \frac{D_{16}}{(D_{11}^3D_{22})^{1/4}} \tag{11c}$$

$$\delta_b = \frac{D_{26}}{(D_{11}D_{22}^3)^{1/4}} \tag{11d}$$

$$K_1 = \frac{N_1^0L_2^2}{\pi^2\sqrt{D_{11}D_{22}}} \tag{11e}$$

$$K_2 = \frac{N_2^0L_1^2}{\pi^2\sqrt{D_{11}D_{22}}} \tag{11f}$$

$$K_{12} = \frac{N_{12}^0L_2^2}{\pi^2(D_{11}D_{22}^3)^{1/4}}. \tag{11g}$$

The parameters α_b, β, γ_b, and δ_b are parameters used to characterize plate buckling (see Stein, 1982; Nemeth, 1986). Similarly, the nondimensional functions K_1, K_2, and K_{12} give rise to the usual definitions of buckling coefficients.

To obtain nondimensional equilibrium and compatibility equations of order one, a new stress resultant function defined by

$$F = \Phi/\sqrt{D_{11}D_{22}} \tag{12}$$

is introduced into Eqs. (8) and (10). Equilibrium Eq. (10) becomes

$$\alpha_b^2 \frac{\partial^4 w}{\partial S_1^4} + 4\alpha_b\gamma_b \frac{\partial^4 w}{\partial S_1^3\partial S_2} + 2\beta \frac{\partial^4 w}{\partial S_1^2\partial S_2^2} + 4\frac{\delta_b}{\alpha_b}\frac{\partial^4 w}{\partial S_1\partial S_2^3} + \frac{1}{\alpha_b^2}\frac{\partial^4 w}{\partial S_2^4} + \frac{L_1^2}{R_1}\frac{\partial^2 F}{\partial S_2^2} + \frac{L_2^2}{R_2}\frac{\partial^2 F}{\partial S_1^2} - K_1\pi^2\frac{\partial^2 w}{\partial S_1^2} - K_2\pi^2\frac{\partial^2 w}{\partial S_2^2} - 2\frac{K_{12}\pi^2}{\alpha_b}\frac{\partial^2 w}{\partial S_1\partial S_2}\ 0. \tag{13}$$

Similarly, substituting Eq. (12) into compatibility Eq. (8) and simplifying gives

$$\alpha_m^2 \frac{\partial^4 F}{\partial S_1^4} + 2\alpha_m\gamma_m \frac{\partial^4 F}{\partial S_1^3\partial S_2} + 2\mu \frac{\partial^4 F}{\partial S_1^2\partial S_2^2} + 2\frac{\delta_m}{\alpha_m}\frac{\partial^4 F}{\partial S_1\partial S_2^3} + \frac{1}{\alpha_m^2}\frac{\partial^4 F}{\partial S_2^4} = \frac{1}{\sqrt{a_{11}a_{22}D_{11}D_{22}}}\left[\frac{L_2^2}{R_2}\frac{\partial^2 w}{\partial S_1^2} + \frac{L_1^2}{R_1}\frac{\partial^2 w}{\partial S_2^2}\right]. \tag{14}$$

A factor of the right-hand-side of Eq. (14) is given by

$$\frac{1}{(a_{11}a_{22}D_{11}D_{22})^{1/4}}$$

and has dimension $1/t$, where t is the shell wall thickness. To put Eq. (14) into a form of order one, a nondimensional buckling displacement W is introduced; i.e.,

$$W = \frac{w}{(a_{11}a_{22}D_{11}D_{22})^{1/4}}. \tag{15}$$

The nondimensional displacement W defined in Eq. (15) has a character that is similar to w/t. Using Eq. (15), Eq. (14) simplifies to

$$\alpha_m^2 \frac{\partial^4 F}{\partial S_1^4} + 2\alpha_m\gamma_m \frac{\partial^4 F}{\partial S_1^3\partial S_2} + 2\mu \frac{\partial^4 F}{\partial S_1^2\partial S_2^2} + 2\frac{\delta_m}{\alpha_m}\frac{\partial^4 F}{\partial S_1\partial S_2^3} + \frac{1}{\alpha_m^2}\frac{\partial^4 F}{\partial S_2^4} = \sqrt{12}\left\{Z_2\frac{\partial^2 W}{\partial S_1^2} + Z_1\frac{\partial^2 W}{\partial S_2^2}\right\} \tag{16}$$

where

$$Z_1 = \frac{L_1^2}{R_1\sqrt{12}(a_{11}a_{22}D_{11}D_{22})^{1/4}} \tag{17a}$$

$$Z_2 = \frac{L_2^2}{R_2\sqrt{12}(a_{11}a_{22}D_{11}D_{22})^{1/4}}. \tag{17b}$$

The term $\sqrt{12}$ is factored out of the right-hand side of Eq. (16) to cancel out a $1/\sqrt{12}$ term that arises when Eqs. (17) are specialized to isotropic shells. Expressions for Z_1 and Z_2 written in terms of shell membrane and bending stiffness coefficients are presented in the Appendix. Equilibrium Eq. (13) also simplifies to

$$\alpha_b^2 \frac{\partial^4 W}{\partial S_1^4} + 4\alpha_b\gamma_b \frac{\partial^4 W}{\partial S_1^3\partial S_2} + 2\beta \frac{\partial^4 W}{\partial S_1^2\partial S_2^2} + 4\frac{\delta_b}{\alpha_b}\frac{\partial^4 W}{\partial S_1\partial S_2^3} + \frac{1}{\alpha_b^2}\frac{\partial^4 W}{\partial S_2^4} + \sqrt{12}\left[Z_1\frac{\partial^2 F}{\partial S_2^2} + Z_2\frac{\partial^2 F}{\partial S_1^2}\right] - K_1\pi^2\frac{\partial^2 W}{\partial S_1^2} - K_2\pi^2\frac{\partial^2 W}{\partial S_2^2} - 2\frac{K_{12}\pi^2}{\alpha_b}\frac{\partial^2 W}{\partial S_1\partial S_2} = 0. \tag{18}$$

Compatibility Eq. (16) can be expressed in operator form as

$$D_m(F) - \sqrt{12}D_c(W) = 0 \tag{19}$$

where $D_m(\ )$ and $D_c(\ )$ are membrane and shell curvature operators, respectively, given by

$$D_m(F) \equiv \alpha_m^2 \frac{\partial^4 F}{\partial S_1^4} + 2\alpha_m \gamma_m \frac{\partial^4 F}{\partial S_1^3 \partial S_2} + 2\mu \frac{\partial^4 F}{\partial S_1^2 \partial S_2^2} + 2\frac{\delta_m}{\alpha_m}\frac{\partial^4 F}{\partial S_1 \partial S_2^3} + \frac{1}{\alpha_m^2}\frac{\partial^4 F}{\partial S_2^4} \quad (20)$$

and

$$D_c(W) \equiv Z_2 \frac{\partial^2 W}{\partial S_1^2} + Z_1 \frac{\partial^2 W}{\partial S_2^2}. \quad (21)$$

The membrane operator given above simplifies to a nondimensional form of the biharmonic operator for isotropic shells.

To put equilibrium Eq. (18) into a useful form, the nondimensional functions K_1, K_2, and K_{12} are expressed in terms of a loading parameter $\tilde{p}$. These relationships are given by

$$K_1 = -P_1 g_1(S_1, S_2)\tilde{p} \quad (22a)$$

$$K_2 = -P_2 g_2(S_1, S_2)\tilde{p} \quad (22b)$$

$$K_{12} = P_3 g_3(S_1, S_2)\tilde{p} \quad (22c)$$

where the minus signs are introduced to make compression loads correspond to positive eigenvalues. The parameters P_1, P_2, and P_3 are load factors that indicate the relative magnitudes of the nondimensional membrane stress resultants prior to buckling, and the functions $g_1(S_1, S_2)$ through $g_3(S_1, S_2)$ indicate the corresponding spatial variations. Using these relations, Eq. (18) is expressed in operator form as

$$D_b(W) + \sqrt{12}D_c(F) = \tilde{p}K_g(W) \quad (23)$$

where $D_b(\ )$ is a bending operator and $K_g(\ )$ is a geometric stiffness operator defined by

$$D_b(W) \equiv \alpha_b^2 \frac{\partial^4 W}{\partial S_1^4} + 4\alpha_b \gamma_b \frac{\partial^4 W}{\partial S_1^3 \partial S_2} + 2\beta \frac{\partial^4 W}{\partial S_1^2 \partial S_2^2} + 4\frac{\delta_b}{\alpha_b}\frac{\partial^4 W}{\partial S_1 \partial S_2^3} + \frac{1}{\alpha_b^2}\frac{\partial^4 W}{\partial S_2^4} \quad (24)$$

$$K_g(W) \equiv -P_1 g_1(S_1, S_2)\pi^2 \frac{\partial^2 W}{\partial S_1^2} - P_2 g_2(S_1, S_2)\pi^2 \frac{\partial^2 W}{\partial S_2^2} + 2P_3 g_3(S_1, S_2)\frac{\pi^2}{\alpha_b}\frac{\partial^2 W}{\partial S_1 \partial S_2}. \quad (25)$$

The bending operator given above also simplifies to a nondimensional form of the biharmonic operator for isotropic shells.

Equations (19) and (23) and the corresponding homogeneous boundary conditions constitute an eigenvalue problem. The smallest value of the loading parameter $\tilde{p}$ for which the equations are satisfied constitutes buckling of the shell. The equations are nondimensional and each derivative is typically of order one for buckle patterns that do not exhibit severe gradients. Thus, the magnitude of the parameters multiplying each derivative term is often a direct indication of the importance of that term to the shell response. Moreover, the parameters defined by Eqs. (11*a*) and (11*b*) and those defined by Eqs. (11*c*) and (11*d*) characterize shell bending orthotropy and bending anisotropy, respectively. Likewise, the parameters defined by Eqs. (9) characterize shell membrane orthotropy and anisotropy. The parameters Z_1 and Z_2 are analogues of the Batdorf Z parameter (Batdorf, 1947a–c) that has long been used to characterize the effects of shell curvature on buckling of isotropic cylindrical shells. The identification of these generalized shell curvature parameters in the present paper was influenced to a great extent by Stein's work (1982), and thus are referred to herein as the Batdorf-Stein Z parameters.

Analogues of Donnell's and Batdorf's Equations

Donnell showed that a single eighth-order differential equation could be obtained for isotropic cylindrical shells by eliminating the stress function appearing in the buckling equations (see Batdorf, 1947a). Applying Donnell's approach to the equations derived herein, Eq. (19) is operated on by the curvature operator $D_c(\ )$ to give

$$D_m(D_c(F)) = \sqrt{12}D_c^2(W) \quad (26)$$

where the order of the operators has been exchanged, in accordance with the commutative property of linear operators with constant coefficients. Next, operating on Eq. (23) with the membrane stiffness operator yields

$$D_m(D_b(W)) + \sqrt{12}D_m(D_c(F)) = \tilde{p}D_m K_g(W). \quad (27)$$

Substituting the right-hand side of Eq. (26) into Eq. (27) yields an eighth-order partial differential equation referred to herein as the Donnell-Stein equation; i.e.,

$$D_m(D_b(W)) + 12D_c^2(W) = \tilde{p}D_m K_g(W). \quad (28)$$

This equation simplifies to a nondimensional equivalent of Donnell's equation for isotropic shells.

Batdorf (1947b,c, 1969) presented an alternative to Donnell's equation that made use of inverse differential operators. The equivalent of Batdorf's equation is obtained by expressing Eq. (19) as

$$F = \sqrt{12}D_m^{-1}(D_c(W)) \quad (29)$$

where $D_m^{-1}(\ )$ denotes the inverse differential operator. Commuting the order of the linear constant coefficient operators in Eq. (29) and substituting the resulting equation into Eq. (23) yields the desired equation; i.e.,

$$D_b(W) + 12D_c^2[D_m^{-1}(W)] - \tilde{p}K_g(W) = 0. \quad (30)$$

This equation is an analogue of Batdorf's modified equilibrium equation (Batdorf, 1947b) and is referred to herein as the modified Batdorf-Stein equation. Equation (30) also simplifies to a nondimensional equivalent of Batdorf's modified equation for isotropic shells.

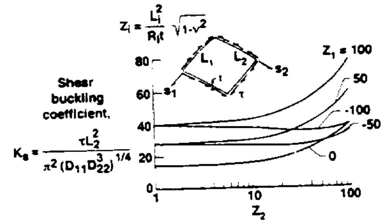

Fig. 2 Effect of shell curvature on shear buckling resistance of isotropic shallow shells ($L_1/L_2 = 1$)

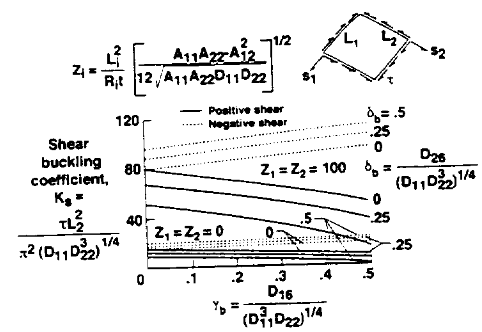

Fig. 3 Effects of bending anisotropy and shell curvature on shear buckling resistance ($L_1/L_2 = 1$)

Selected Results and Discussion

Some selected results are presented in Figs. 2 and 3 for shear buckling of balanced symmetrically laminated shallow shells to demonstrate the usefulness of the nondimensional parameters identified herein. The results were obtained from a Bubnov-Galerkin approximate solution to the modified Batdorf-Stein equation given by Eq. (30). Details of the analytical solution have been presented by Nemeth (1991). A typical shell is loaded on its edges by a uniform tangential shearing traction τ as shown in Fig. 2. For this loading, $N_{12}^0(s_1, s_2) = \tau$ is the only nonzero prebuckling stress resultant. The shell is supported such that the buckling displacement w and the rotations $\partial w/\partial s_1$ and $\partial w/\partial s_2$ vanish along all the edges of the shell. In addition, the membrane displacements normal to each edge of the shell are restrained and no additional tangential shearing tractions occur due to buckling. The buckling resistance of a shell is indicated in the figures by the nondimensional shear buckling coefficient K_s defined by

$$K_s = \frac{\tau L_2^2}{\pi^2 (D_{11} D_{22}^3)^{1/4}} \tag{31}$$

where τ is the applied uniform shearing traction. For the results presented in the figures, the orthotropic parameters (α_b, β, α_m, and μ) and the anisotropic parameters (γ_b, δ_b, γ_m, and δ_m) that are not varied are set equal to one and zero, respectively. This baseline set of values corresponds to an isotropic shell with sides of equal length ($L_1 = L_2$).

Results showing shear buckling coefficient as a function of the Batdorf-Stein shell curvature parameters, Z_1 and Z_2, are presented in Fig. 2. The abscissa of the plot shown in Fig. 2 is measured by a logarithmic scale. Results are shown in this figure for $Z_1 = 0$ and $Z_2 = 1$, which correspond essentially to flat plates, and for shells with zero ($Z_1 Z_2 = 0$), negative ($Z_1 Z_2 < 0$), and positive ($Z_1 Z_2 > 0$) Gaussian curvature with magnitudes of Z_1 and Z_2 ranging from 0 to 100. The analytical results presented in Fig. 2 predict that the shear buckling resistance of a shell is significantly influenced by shell curvature, especially for the larger values of Z_1 and Z_2 shown in the figure. The results also indicate that the shells with positive Gaussian curvature are more buckling resistant than those with negative Gaussian curvature, which, in turn, are typically more buckling resistant than those with zero Gaussian curvature. Flat plates exhibit the lowest buckling resistance.

Results showing shear buckling coefficient as a function of the curvature parameters, Z_1 and Z_2, and the bending anisotropy parameters γ_b and δ_b are presented in Fig. 3. Results are shown in this figure for flat plates ($Z_1 = Z_2 = 0$) and for shells with positive Gaussian curvature corresponding to $Z_1 = Z_2 = 100$. Values of the anisotropic parameters range from 0 to 0.5. This range of values is considered to be representative of a large class of laminated plates (Nemeth, 1986). Results are presented in Fig. 3 corresponding to both positive (as shown in the figure) and negative directions of the applied shearing traction, in accordance with the presence of bending anisotropy.

The results presented in Fig. 3 predict that the shear buckling resistance of a shell with positive Gaussian curvature is more sensitive to variations in the anisotropic (bending) parameters than a corresponding flat plate. The results show substantial reductions in buckling resistance with increasing values of the anisotropic parameters for shells loaded in positive shear, and similar increases in buckling resistance for shells loaded in negative shear. Similar results were obtained for a corresponding shell with negative Gaussian curvature that indicate the same trend, but not to as large as extent as exhibited by the shell with positive Gaussian curvature.

The generic results presented in Figs. 2 and 3 apply to many laminate constructions, and show that varying parameters independently can give insight into the factors strongly affecting the structural response. For example, by independently varying the parameters associated with shell curvature, positive values of Gaussian curvature are found to improve substantially the shear buckling resistance of a shell. In addition, shell curvature was determined to affect significantly the importance of the bending anisotropy on the shear buckling resistance. Both of these observations clearly indicate the benefits of using nondimensional parameters to formulate the analysis and to perform parametric studies.

Concluding Remarks

A method of deriving nondimensional equations and identifying the fundamental parameters associated with bifurcation buckling of shallow, anisotropic shells subjected to combined loads has been presented. Specifically, analysis has been presented for symmetrically laminated doubly curved shells that exhibit both membrane and bending anisotropy, and the procedure and rationale required to obtain useful nondimensional forms of the transverse equilibrium and compatibility equations for buckling have been discussed. The analysis presented herein yields fundamental parameters that explicitly and compactly indicate the effects of both membrane and bending orthotropy and anisotropy, and the effects of curvature on shallow shell buckling behavior. An important contribution of this work is the development of anisotropic analogues of the well-known Batdorf Z shell curvature parameter for symmetrically laminated anisotropic shells with compound curvature, and corresponding analogues of Donnell's and Batdorf's shell buckling equations.

Results obtained from a Bubnov-Galerkin solution to a representative example problem have also been presented. The results indicate the utility of recasting the shell buckling equations in terms of nondimensional parameters to conduct parametric studies that are generic and well suited to the preliminary design of laminated composite shells. Moreover, the analytical results predict that shells with positive Gaussian curvature are much more resistant to shear buckling than corresponding flat plates or shells with negative and zero Gaussian curvature. In addition, the results predict that the importance of bending anisotropy on shear buckling resistance is significantly affected by shell curvature.

References

Almroth, B. O., and Brogan, F. A., 1972, "Bifurcation Buckling as an Approximation of the Collapse Load for General Shells," *AIAA Journal*, Vol. 10, No. 4, pp. 463–467.

Batdorf, S. B., 1947a, "A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells, I—Donnell's Equation," NACA Technical Note No. 1341.

Batdorf, S. B., 1947b, "A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells, II—Modified Equilibrium Equation," NACA Technical Note No. 1342.

Batdorf, S. B., 1947c, "A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells," NACA Technical Report No. 874.

Batdorf, S. B., 1969, "On the Application of Inverse Differential Operators to the Solution of Cylinder Buckling and Other Problems," *Proceedings of the AIAA/ASME 10th Structures, Structural Dynamics, and Materials Conference*.

Dong, S. B., Pister, K. S., and Taylor, R. L., 1962, "On the Theory of Laminated Anisotropic Shells and Plates," *Journal of the Aerospace Sciences*, Vol. 29, pp. 969–975.

Gerard, G., 1962, *Introduction to Structural Stability Theory*, McGraw-Hill, New York, pp. 129–155.

Nemeth, M. P., 1986, "Importance of Anisotropy on Buckling of Compression-Loaded Symmetric Composite Plates," *AIAA Journal*, Vol. 24, No. 11, pp. 1831–1835.

Nemeth, M. P., 1991, "Nondimensional Parameters and Equations for Buckling of Symmetrically Laminated Thin Elastic Shallow Shells," NASA Technical Memorandum 104060.

Stein, M., and McElman, J. A., 1965, "Buckling of Segments of Toroidal Shells," *AIAA Journal*, Vol. 3, No. 9, pp. 1704–1709.

Stein, M., 1982, "Postbuckling of Orthotropic Composite Plates Loaded in Compression," Presented at the AIAA/ASME/ASCE/AHS 23rd Structures, Structural Dynamics, and Materials Conference, AIAA Paper No. 82-0778, New Orleans, LA.

APPENDIX

Parameters in Terms of Membrane Stiffnesses

The parameters α_m, μ, γ_m, δ_m, Z_1, and Z_2 have been given in the present paper in terms of membrane flexibility coefficients. More convenient forms of these parameters are obtained by expressing them in terms of the membrane stiffness coefficients. Inverting the membrane stiffness matrix $[A]$ associated with Eqs. (6) and substituting the resulting expressions into the expressions for the nondimensional parameters gives

$$\alpha_m = \frac{L_2}{L_1}\left(\frac{A_{11}A_{66}-A_{16}^2}{A_{22}A_{66}-A_{26}^2}\right)^{1/4} \tag{A1}$$

$$\mu = \frac{A_{11}A_{22}-A_{12}^2-2A_{12}A_{66}+2A_{16}A_{26}}{2[(A_{11}A_{66}-A_{16}^2)(A_{22}A_{66}-A_{26}^2)]^{1/2}} \tag{A2}$$

$$\gamma_m = \frac{A_{11}A_{26}-A_{12}A_{16}}{[(A_{11}A_{66}-A_{16}^2)^3(A_{22}A_{66}-A_{26}^2)]^{1/4}} \tag{A3}$$

$$\delta_m = \frac{A_{22}A_{16}-A_{12}A_{26}}{[(A_{11}A_{66}-A_{16}^2)(A_{22}A_{66}-A_{26}^2)^3]^{1/4}} \tag{A4}$$

$$Z_i = \frac{L_i^2}{R_i}\left[\frac{(A_{11}A_{22}-A_{12}^2)A_{66}-A_{11}A_{26}^2-A_{22}A_{16}^2+2A_{12}A_{16}A_{26}}{12[(A_{11}A_{66}-A_{16}^2)(A_{22}A_{66}-A_{26}^2)D_{11}D_{22}]^{1/2}}\right]^{1/2} \tag{A5}$$

where i appearing in Eq. (A5) is a free index that takes on the values of 1 and 2. For balanced symmetric laminates with A_{16} and A_{26} that are zero valued, $\gamma_m = 0$ and $\delta_m = 0$. The remaining nonzero parameters simplify to

$$\alpha_m = \frac{L_2}{L_1}(A_{11}/A_{22})^{1/4} \tag{A6}$$

$$\mu = \frac{A_{11}A_{22}-A_{12}^2-2A_{12}A_{66}}{2A_{66}\sqrt{A_{11}A_{22}}} \tag{A7}$$

$$Z_i = \frac{L_i^2}{R_i}\left[\frac{A_{11}A_{22}-A_{12}^2}{12\sqrt{A_{11}A_{22}D_{11}D_{22}}}\right]^{1/2}. \tag{A8}$$

In addition, when the bending stiffnesses D_{16} and D_{26} are zero valued, $\gamma_b = \delta_b = 0$. Similarly, for isotropic shells, the nonzero parameters are given by

$$\alpha_m = \alpha_b = \frac{L_2}{L_1} \tag{A9}$$

$$\mu = \beta = 1 \tag{A10}$$

$$Z_i = \frac{L_i^2}{R_i t}(1-\nu^2)^{1/2} \tag{A11}$$

where t is the total shell thickness and ν is Poisson's ratio.